Numerical Methods for Geometric Motion

Brian Wetton Keith Promislow

Mathematics Department, UBC
www.math.ubc.ca/∼wetton

ACM Reunion

Institute of Applied Mathematics

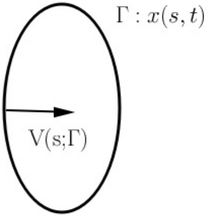
University of British Columbia

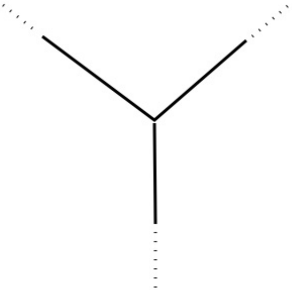
- Faculty participation from many departments.
- Interdisciplinary graduate programme.

Overview of the Talk

- Geometric Motion (2D Curvature Motion Example)
- Numerical Method (Formulation)
- Gradient Flow Dynamics

Geometric Motion - I





Examples:

- $V = \kappa$ (curvature)
- $V = -\kappa_{ss}$ (surface diffusion)
- Mullins-Sekerka (nonlocal)

Numerical Challenges:

- Topological changes
- Viscosity solutions
- Networks with junctions
- Stiff systems

Applications:

- Image processing
- Materials Science
- (Intrinsic Interest)

Geometric Motion-II

- "Geometric" means that the dynamics only depends on the curve shape.
- Only the normal velocity is needed to specify the dynamics.
- We will consider first 2D curvature motion of a simple, closed curve, $V = \kappa$.

Gradient Flow - I

Allen-Cahn dynamics are a gradient flow on the energy functional

$$\mathcal{E}(u) = \int \left(\frac{\epsilon^2}{2} |\nabla u|^2 + W(u) \right).$$

This can be seen by calculating (integrate by parts)

$$\frac{d\mathcal{E}}{dt} = \int u_t \left(-\epsilon^2 \Delta u + W'(u) \right).$$

Taking the dynamics to be

$$u_t = \epsilon^2 \Delta u - W'(u)$$

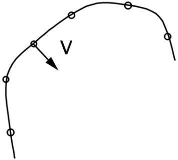
makes

$$\frac{d\mathcal{E}}{dt} \leq 0.$$

Gradient Flow - II

- Curvature motion inherits a gradient flow nature from Allen-Cahn.
- Energy $\mathcal{E} = L$ (curve length).
- Gradient flow
$$\frac{d\mathcal{E}}{dt} = -\int_{\Gamma} \kappa^2.$$
- Main Idea: Derive and compute gradient flows directly from intrinsic curve energies.

Curve tracking $x(\sigma, t)$ formulation



- $x(\sigma, t)$, with $x_t \cdot \hat{n} = V$
- Tangential velocity maintains scaled arc-length, impose this directly:

$$\frac{1}{2}\frac{\partial}{\partial \sigma}|x_\sigma|^2 = x_\sigma \cdot x_{\sigma\sigma} = 0 \quad \text{or} \quad |x_\sigma| = L$$

- Fix arbitrary constant in tangential velocity:

$$\int_0^1 x_t \cdot \hat{\tau} d\sigma = 0$$

- Curvature $\kappa = x_{\sigma\sigma} \cdot (x_\sigma)^\perp / L^3$
- FD discretization, implicit time stepping (index-1 DAE structure). Highly stiff and nonlinear dynamics

Gradient Flow for Curvature Motion

$$
\begin{aligned}
L^2 &= \int_0^1 |x_\sigma|^2 d\sigma \\
2L\frac{dL}{dt} &= 2\int_0^1 x_\sigma \cdot x_{t\sigma} d\sigma \\
\dot{L} &= 1/L^2 \int_0^1 x_\sigma \cdot (V x_\sigma^\perp + W x_\sigma)_\sigma d\sigma \\
\dot{L} &= 1/L^2 \int_0^1 \left\{ -V(x_{\sigma\sigma} \cdot x_\sigma^\perp) + W x_\sigma \cdot x_{\sigma\sigma} + W_\sigma x_\sigma \cdot x_\sigma \right\} d\sigma \\
\dot{L} &= -L \int_0^1 V \kappa d\sigma
\end{aligned}
$$

Gradient flow is $V = \kappa$.

Gradient Flow for other $\mathcal{E}(\Gamma)$

Adhesion Energy

- Adhesion Energy

$$\mathcal{E} = L \int_0^1 \frac{1}{2}\kappa^2 \, d\sigma + L^2 \int_0^1 \int_0^1 G(|d|^2) \, d\sigma \, d\sigma'$$

- G has a minimum at a prescribed distance.
- Gradient flow velocity Promislow:

$$V = \left(\Delta_s + \frac{\kappa^2}{2} - \mathbb{B}(\sigma) \right) \kappa - \mathbb{A}(\sigma) \cdot n(s).$$

where

$$\mathbb{A}(\sigma) := 4L \int_0^1 2G'(|d|^2); d\sigma'$$

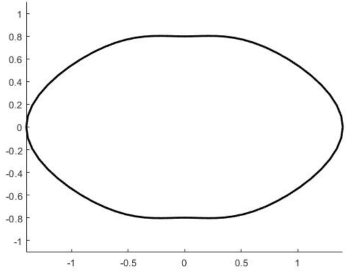
and

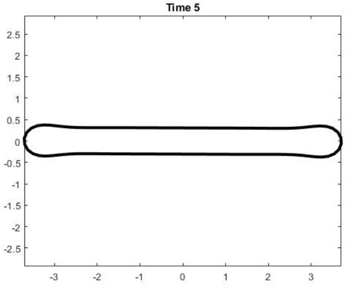
$$\mathbb{B}(\sigma) := 2L \int_0^1 G(|d|^2); d\sigma'$$

- x tracking results movie.

Adhesion Energy

Results





General x Code

- Ongoing (just started) project for an open source code that can handle a variety of (local) geometric motion velocities.
- Up to sixth order terms (fourth order in curvature, regularization of faceting curvature terms).
- Include scalar concentration dynamics on the curve that influences normal motion.
- Adaptive implicit time stepping.
- Direct solves for Newton iterations.

Summary

- Considered gradient flows on intrinsic curve energies.
- Proposed open source framework to handle a general class of 2D local geometric motion problems.