Numerical Methods for Geometric Motion

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ACM Reunion
Institute of Applied Mathematics
University of British Columbia

- Faculty participation from many departments.
- Interdisciplinary graduate programme.
Overview of the Talk

- Geometric Motion (2D Curvature Motion Example)
- Numerical Method (Formulation)
- Gradient Flow Dynamics
Geometric Motion - I

Examples:
- \( V = \kappa \) (curvature)
- \( V = -\kappa_{ss} \) (surface diffusion)
- Mullins-Sekerka (nonlocal)

Numerical Challenges:
- Topological changes
- Viscosity solutions
- Networks with junctions
- Stiff systems

Applications:
- Image processing
- Materials Science
- (Intrinsic Interest)
Geometric Motion-II

• “Geometric” means that the dynamics only depends on the curve shape.
• Only the normal velocity is needed to specify the dynamics.
• We will consider first 2D curvature motion of a simple, closed curve, $V = \kappa$. 
Gradient Flow - 1

Allen-Cahn dynamics are a gradient flow on the energy functional

$$\mathcal{E}(u) = \int \left( \frac{\epsilon^2}{2} |\nabla u|^2 + W(u) \right) .$$

This can be seen by calculating (integrate by parts)

$$\frac{d\mathcal{E}}{dt} = \int u_t \left( -\epsilon^2 \Delta u + W'(u) \right) .$$

Taking the dynamics to be

$$u_t = \epsilon^2 \Delta u - W'(u)$$

makes

$$\frac{d\mathcal{E}}{dt} \leq 0 .$$
Gradient Flow - II

- Curvature motion inherits a gradient flow nature from Allen-Cahn.
- Energy $\mathcal{E} = L$ (curve length).
- Gradient flow
  $$\frac{d\mathcal{E}}{dt} = -\int_{\Gamma} \kappa^2.$$
- Main Idea: Derive and compute gradient flows directly from intrinsic curve energies.
Curve tracking $x(\sigma, t)$ formulation

- $x(\sigma, t)$, with $x_t \cdot \hat{n} = V$
- Tangential velocity maintains scaled arc-length, impose this directly:
  \[
  \frac{1}{2} \frac{\partial}{\partial \sigma} |x_\sigma|^2 = x_\sigma \cdot x_{\sigma \sigma} = 0 \quad \text{or} \quad |x_\sigma| = L
  \]
- Fix arbitrary constant in tangential velocity:
  \[
  \int_0^1 x_t \cdot \hat{\tau} d\sigma = 0
  \]
- Curvature $\kappa = x_{\sigma \sigma} \cdot (x_\sigma)^\perp / L^3$
- FD discretization, implicit time stepping (index-1 DAE structure). Highly stiff and nonlinear dynamics
Gradient Flow for Curvature Motion

\[ L^2 = \int_0^1 |x_\sigma|^2 d\sigma \]

\[ 2L \frac{dL}{dt} = 2 \int_0^1 x_\sigma \cdot x_{t\sigma} d\sigma \]

\[ \dot{L} = \frac{1}{L^2} \int_0^1 x_\sigma \cdot (Vx_\sigma^\perp + Wx_\sigma) d\sigma \]

\[ \dot{L} = \frac{1}{L^2} \int_0^1 \left\{ -V(x_{\sigma\sigma} \cdot x_\sigma) + Wx_\sigma \cdot x_{\sigma\sigma} + W_\sigma x_\sigma \cdot x_\sigma \right\} d\sigma \]

\[ \dot{L} = -L \int_0^1 V \kappa d\sigma \]

Gradient flow is \( V = \kappa \).
Gradient Flow for other $\mathcal{E}(\Gamma)$

Adhesion Energy

- Adhesion Energy
\[ \mathcal{E} = L \int_0^1 \frac{1}{2} \kappa^2 \, d\sigma + L^2 \int_0^1 \int_0^1 G(|d|^2) \, d\sigma \, d\sigma' \]

- $G$ has a minimum at a prescribed distance.
- Gradient flow velocity **Promislow**:
\[ V = \left( \Delta_s + \frac{\kappa^2}{2} - B(\sigma) \right) \kappa - A(\sigma) \cdot n(s). \]

where
\[ A(\sigma) := 4L \int_0^1 2G'(|d|^2); d\sigma' \]
and
\[ B(\sigma) := 2L \int_0^1 G(|d|^2); d\sigma' \]

- x tracking results **movie.**
Adhesion Energy

Results
General x Code

- Ongoing (just started) project for an open source code that can handle a variety of (local) geometric motion velocities.
- Up to sixth order terms (fourth order in curvature, regularization of faceting curvature terms).
- Include scalar concentration dynamics on the curve that influences normal motion.
- Adaptive implicit time stepping.
- Direct solves for Newton iterations.
Summary

- Considered gradient flows on intrinsic curve energies.
- Proposed open source framework to handle a general class of 2D local geometric motion problems.