A Nontraditional Regularity Criterion for the 3D Navier-Stokes Equations

BY XINWEI YU

University of Alberta

Nov. 12, 2023

International Workshop on Recent Developments in Applied Mathematics and its Applications
Caltech -- Pasadena, California

Joint work with Chuong V. Tran, University of St. Andrews, UK

Plan of the Talk

1. Introduction
    a) The incompressible Navier-Stokes equations;
    b) The well-posedness problem;
    c) Regularity criteria.
2. Main Result: Statement and Discussions.
3. Proof of Main Result.
    a) Pressure moderation;
    b) Energy estimate;
    c) Instantaneous decay;
    d) Proof of the main result.

Introduction: The Navier-Stokes Equations

- 3D NSE on $\mathbb{R}^3 \times [0, T)$.

$$
\begin{aligned}
u_t + (u \cdot \nabla)u &= -\nabla p + \nu \Delta u, &(1)\\
\nabla \cdot u &= 0, &(2)\\
u(x, 0) &= u_0(x). &(3)
\end{aligned}
$$

$x \in \mathbb{R}^3$, $t \in [0, T)$.

- Will set $\nu = 1$ in the following.
- Leray 1934.
  - For $u_0 \in L^2$ there is a weak solution $u \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$ for any $T > 0$.
  - For $u_0 \in H^1 \cap C^1$ there is $T_0 > 0$ and a unique classical solution on $[0, T_0)$.

Introduction: The Well-posedness Problem

- Many improvements and generalizations (Kato 1972, Caffarelli-Kohn-Nirenberg 1982, Giga 1986, ...).
- Basic situation remains:
  - Weak solutions exist globally in time with uniqueness unknown;
  - Classical solutions are unique but their global existence has not been proved.
- The questions.
  - Are weak solutions unique?
  - Does smooth solution exist globally in time?
- Global regularity only established under additional hypotheses (regularity criteria).

Introduction: Classical Regularity Criteria

- Prodi-Serrin-Ladyzhenskaya: Weak solutions are smooth on $(0,T]$ (and thus can be extended beyond T) under the following additional assumption:

$$\int_0^T \|u(t)\|_{L^s}^r \, \mathrm{d}t < \infty, \qquad s \in [3,\infty],\ r > 0,\ \frac{3}{s} + \frac{2}{r} \leqslant 1. \tag{4}$$

  $s = 3$: Escauriaza-Seregin-Sverak 2003;

- PSL-type criteria for pressure:
  - Observation: $\nabla \cdot [u_t + u \cdot \nabla u + \nabla p - \nu \Delta u] = 0 \Longrightarrow$

$$-\Delta p = \nabla \cdot [\nabla \cdot (u \otimes u)] \Longrightarrow p \sim u^2. \tag{5}$$

  - Calderon-Zygmund inequality: For $\gamma \in (1,\infty)$,

$$\|p\|_{L^\gamma} \lesssim \|u^2\|_{L^\gamma} = \|u\|_{L^{2\gamma}}^2. \tag{6}$$

  - Expect criteria of the form $p \in L^r(0,T;L^s)$ with $s > \frac{3}{2}$ and $\frac{3}{s} + \frac{2}{r} \leqslant 2$. (Chae-Lee 2001, Berselli 2002)

Introduction: Other Regularity Criteria

- Generalizations of PSL.
  - Replace $L^s(L^r)$ by weaker spaces. (Sohr 2001, Chen-Zhang 2007, Fan-Jiang-Ni 2008, Ben-bernou 2009, He-Gala 2011, Zhang-Jia-Dong 2012, Bosia-Pata-Robinson 2014, ...)
  - Weaken by a log factor $(1+\log\|u\|)^{-\alpha}$. (Chan-Vasseur 2007, Zhou 2009, Zhou-Gala 2009, Fan-Jiang-Nakamura-Zhou 2011, Fan-Jiang-Nakamura 2011; Guo-Gala 2011, Liu-Zhao-Cui 2012, ...)
  - Weaken by a power factor $(1+\|u\|)^{-\kappa}$. (Tran-Yu 2017)
  - Weaken by replacing the full vector u with one component. (Kukavica-Ziane 2006, Zhou-Pokorny 2010, Cao-Titi 2011, Chemin-Zhang 2017, Han-Lei-Li-Zhao 2018, ...)
  - And many more.
- Many other types of criteria. (Constantin-Fefferman 1993, Seregin-Sverak 2002, Grujic-Zhang 2006, Grujic 2009, Vasseur 2009, Chan 2010, Grujic 2013...)

Main Result: Statement

Theorem 1. *Let u solve the NSE and be smooth up to (not including) $T > 0$. Assume*

i. $u_0 \in L^2 \cap L^\infty$;

ii. $\exists Q(t) \in L^\infty_{\text{loc}}[0, T)$, $\forall q > Q(t)$, $\exists f_q(x, t) \in W^{1,\infty}$ *and* $u \cdot \nabla f_q = 0$, *such that* $D(x, t) := \nabla \cdot ((p + f_q)|u|^{q-2} u) \neq 0$ *in* $\Omega(t, q) := \{(x, t) | \, |u| > U(t, q)\}$ *where* $U(t, q) = C(q) \, \|u\|_{L^{3q-6}}$.[1]

Then u remains smooth beyond T.

Remark 2.

- $Q(t)$ does not have to remain bounded as $t \nearrow T$. For example it could be $\frac{1}{T-t}$.
- No assumption on any of the norms of u or p.

1. The constant $C(q)$, independent of t, will be defined later.

Main Result: A Paradigm Scenario

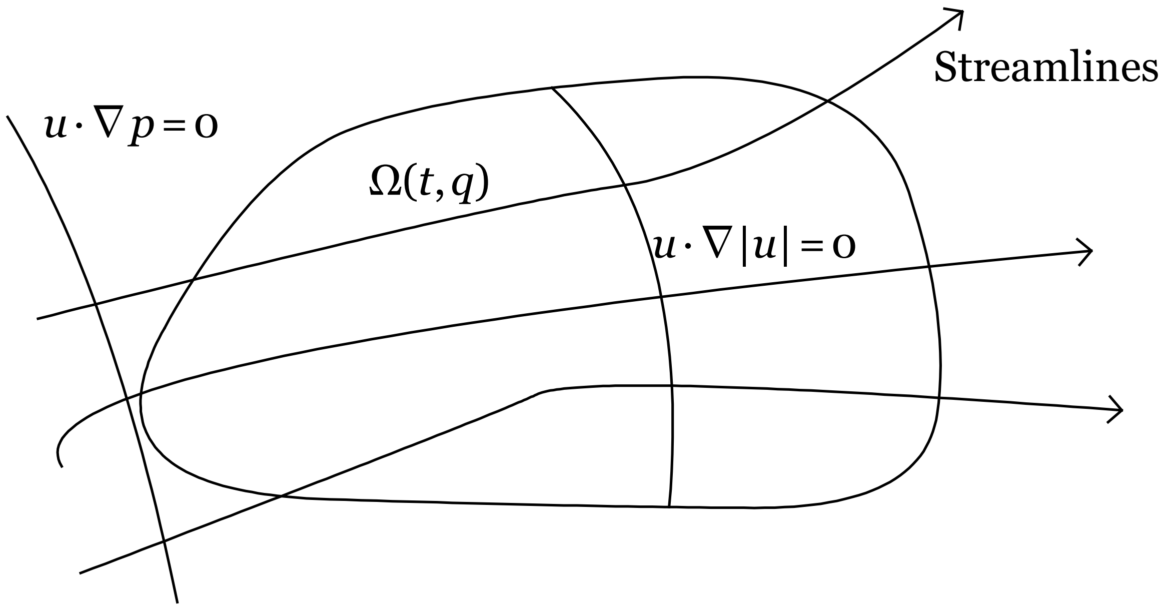

Figure 1. A schematic description of the high velocity region $\Omega(t,q)$. Note that at a fixed time t, the locations of $u \cdot \nabla p = 0$ and $u \cdot \nabla |u| = 0$ are fixed but $\Omega(t,q)$ is shrinking as $q \nearrow \infty$.

Main Result: A Paradigm Scenario

- Assume that the streamlines remain laminar and smooth inside $\Omega(t,q)$;
- $d(\Omega)$: arc length of the longest streamline within Ω;
- $u\cdot\nabla|u| = 0$ and $u\cdot\nabla p = 0$ are smooth enough surfaces, and do not intersect.

Corollary 3. *Under these assumptions, if $q^{1/2}\, d(\Omega(t,q)) \longrightarrow 0$ as $q \nearrow \infty$ at every time $t \in (0,T)$, then the solution remains smooth beyond T.*

- The convergence $q^{1/2}\, d(\Omega(t,q)) \longrightarrow 0$ does not need to be uniform in t.

Proof. Observe $D = |u|^{q-1}\left[u\cdot\nabla p + (q-2)\,(p+f)\,\frac{u\cdot\nabla|u|}{|u|}\right]$.

1. Take $f_q \equiv f$ where $u\cdot\nabla f = 0$ and $f = -p$ along $u\cdot\nabla|u| = 0$;
2. $(q-2)\,(p+f)\,\frac{u\cdot\nabla|u|}{|u|} \leqslant c\,(q-2)\,d(\Omega)^2$ in $\Omega(t,q)$;
3. $|u\cdot\nabla p| \geqslant C$ in $\Omega(t,q)$.

Thus $D \neq 0$ for all q large enough.

Proof: Pressure Moderation

Lemma 4. *Let* $u:\mathbb{R}^3\mapsto\mathbb{R}^3$, $f:\mathbb{R}^3\mapsto\mathbb{R}$, *and* $g:\mathbb{R}^+\cup\{0\}\mapsto\mathbb{R}$. *Assume*

i. $\nabla\cdot u=0$, $u\cdot\nabla f=0$;

ii. $u\in H^1\cap L^2\cap L^\infty$, $f\in W^{1,\infty}$, *and* $g\in L^\infty\cap L_{\text{loc}}$.

Then for arbitrary $q\in\mathbb{N}$, *we have*

$$\int_{\mathbb{R}^3} f(x,t)\,g(|u|)\,|u|^q u\cdot\nabla|u|\,\mathrm{d}x=0. \tag{7}$$

Remark 5. We call $\mathscr{P}(x,t):=f(x,t)\,g(|u|)$ a "pressure moderator".

Proof.

1. Let $H(s):=\int_0^s g(r)\,r^q\,\mathrm{d}r$; Need to show $u\cdot\nabla H(|u|)=g(|u|)\,|u|^q u\cdot\nabla|u|$.
2. Check:
   i. $H(s)$ is Lipschitz;
   ii. $H(|u|)\in L^2$.
3. Apply Theorem 2.1.11 of W. P. Ziemer *Weakly Differentiable Functions*.

Proof: Energy Estimate

- Integration of the momentum equation.

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^q}^q = (q-2)\int_{\mathbb{R}^3} p\,|u|^{q-3}u\cdot\nabla|u|\,\mathrm{d}x$$
$$-(q-2)\,\||u|^{(q-2)/2}\nabla|u|\|_{L^2}^2-\||u|^{(q-2)/2}\nabla u\|_{L^2}^2. \qquad (8)$$

- Localized pressure moderator (The cut-off U will be chosen later)

$$g(s)=\begin{cases} 0 & s\leqslant U \\ 1 & s>U \end{cases} \Longrightarrow \mathscr{P}(x,t):=f(x,t)\,g(|u|)=0 \text{ outside } \Omega:=\{|u|>U\}. \qquad (9)$$

- (8) becomes

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^q}^q = (q-2)\int_{\mathbb{R}^3}(p+\mathscr{P})\,|u|^{q-3}u\cdot\nabla|u|\,\mathrm{d}x$$
$$-(q-2)\,\||u|^{(q-2)/2}\nabla|u|\,\|_{L^2}^2-\||u|^{(q-2)/2}\nabla u\|_{L^2}^2$$
$$= (q-2)\int_{\Omega}(p+f)\,|u|^{q-3}u\cdot\nabla|u|\,\mathrm{d}x$$
$$+(q-2)\int_{\Omega^c} p\,|u|^{q-3}u\cdot\nabla|u|\,\mathrm{d}x$$
$$-(q-2)\,\||u|^{(q-2)/2}\nabla|u|\,\|_{L^2}^2-\||u|^{(q-2)/2}\nabla u\|_{L^2}^2. \qquad (10)$$

Proof: Contribution from Ω^c

- Take

$$U = U(t,q) := C(q)\,\|u\|_{L^{3q-6}}, \qquad C(q) = (c_0\,c_1\,q\,\|u_0\|_{L^2})^{-2/(q-2)} \tag{11}$$

where c_0, c_1 come from the Calderon-Zygmund and Sobolev inequalities

$$\|p\|_{L^2} \leqslant c_0\,\|u\|_{L^4}^2, \qquad \|h\|_{L^6} \leqslant c_1\,\|\nabla h\|_{L^2}. \tag{12}$$

- Set $\Omega = \Omega(t,q) := \{(x,t)|\,|u| > U\}$.

- Contribution outside Ω is then negligible.

$$\int_{\Omega^c} p\,|u|^{q-3}\,u\cdot\nabla|u|\,\mathrm{d}x \leqslant \frac{1}{2}\,\|\,|u|^{(q-2)/2}\nabla|u|\,\|_{L^2}. \tag{13}$$

Proof. Standard interpolation, etc.

Proof: Main Estimate

- Re-formulation.

$$\int_\Omega p\,|u|^{q-3}u\cdot\nabla|u|\,\mathrm{d}x = \int_\Omega (p+f)\,|u|^{q-2}u\cdot\frac{\nabla(|u|/U)}{|u|/U}\,\mathrm{d}x$$
$$= \int_\Omega \ln\left(\frac{U}{|u|}\right)\nabla\cdot[(p+f)\,|u|^{q-2}u]\,\mathrm{d}x. \tag{14}$$

- Dissipation terms.

$$\|\,|u|^{(q-2)/2}\nabla u\|_{L^2}, \|\,|u|^{(q-2)/2}\nabla|u|\,\|_{L^2} \geqslant \frac{2}{q\,c_1}\|u\|_{L^{3q}}^{q/2}. \tag{15}$$

- Main Estimate.

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^q}^q \leqslant q\,(q-2)\int_\Omega \ln\left(\frac{U}{|u|}\right)D(x,t)\,\mathrm{d}x - \frac{2}{c_1^2}\|u\|_{L^{3q}}^q. \tag{16}$$

where

$$D(x,t) := \nabla\cdot[(p+f)\,|u|^{q-2}u]. \tag{17}$$

- Strategy: RHS of (16) <0 for all $q > Q(t)$.
- Problem: $Q(t)$ may be unbounded as $t \nearrow \infty$.

Proof: Instantaneous Decay

Lemma 6. *Let u solve the 3D Navier-Stokes equations that is smooth on* $(0, T)$. *Assume*

i. $u_0 \in L^2 \cap L^\infty$;

ii. $\exists Q(t) \in L^\infty_{\mathrm{loc}}[0, T), \forall t \in (0, T), \forall q > Q(t), \frac{\mathrm{d}}{\mathrm{d}t} \|u\|_{L^q} < 0$.

Then the solution remains smooth beyond T.

Proof. Wlog $\|u_0\|_{L^2} \leqslant 1$. Assume u "blows up" at T.

1. Interpolation $\Longrightarrow \forall q \in [2, \infty], \|u_0\|_{L^q} \leqslant M$ for some constant M;
2. Prodi-Serrin $\Longrightarrow \exists q_0 > 3, \ \|u\|_{L^{q_0}} \longrightarrow \infty$ at T;
3. Blow-up rate: $\|u\|_{L^{q_0}} \geqslant \frac{1}{(T-t)^\beta}$ for $t \in (T - \delta, T)$;
4. Interpolation $\Longrightarrow \ \forall q > q_0, \ \|u\|_{L^q} \geqslant \frac{1}{(T-t)^\beta}$;
5. But $\|u\|_{L^q} \leqslant \|u_0\|_{L^q} \leqslant M$ until $Q(t) \geqslant q$. Contradiction for large q.

Remark 7. There may be no single q such that $\frac{\mathrm{d}}{\mathrm{d}t} \|u\|_{L^q} < 0$ for all $t \in (0, T)$.

Proof of the Theorem

- Recall the main estimate (16).

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^q}^q \leqslant q(q-2)\int_\Omega \ln\left(\frac{U}{|u|}\right) D(x,t)\,\mathrm{d}x - \frac{2}{c_1^2}\|u\|_{L^{3q}}^q. \tag{18}$$

- Strategy: Fix $t \in (0,T)$. Show RHS <0 as $q \nearrow \infty$.
- Assumption $D \neq 0$ in Ω. Thus $D > 0$ or $D < 0$.
  - $D > 0 \Longrightarrow$ RHS <0;
  - Only need to work on $D < 0$.

Proof of the Theorem: The case $D < 0$

1. Mean value theorem. $U_+ \in [U(t,q), \|u\|_{L^\infty}]$.

$$\int_\Omega \ln\left(\frac{U}{|u|}\right) D(x,t)\,\mathrm{d}x \;=\; \ln\frac{U}{U_+}\int_\Omega \nabla\cdot[(p+f)\,|u|^{q-2}u]\,\mathrm{d}x$$

$$= \; U^{q-1}\ln\frac{U}{U_+}\int_{\partial\Omega}(p+f)\,\hat{u}\cdot\mathrm{d}A. \qquad (19)$$

2. Substitute in $U = (c_0\, c_1\, q\,\|u_0\|_{L^2})^{-2/(q-2)}\,\|u\|_{L^{3q-6}}$.

$$\mathrm{RHS} \leqslant \left[\frac{q\,(q-2)}{(c_0^2\,c_1^2\,q^2\,\|u_0\|_{L^2}^2)^{(q-1)/(q-2)}}\ln\frac{U}{U_+}\int_{\partial\Omega}\cdots\mathrm{d}A - \frac{2}{c_1^2}\frac{\|u\|_{L^{3q}}^{q}}{\|u\|_{L^{3q-6}}^{q-1}}\right]\|u\|_{L^{3q-6}}^{q-1} \qquad (20)$$

3. $q \nearrow \infty$: Note that t is fixed here.

$$\lim_{q\to\infty}\frac{q\,(q-2)}{(c_0^2\,c_1^2\,q^2\,\|u_0\|_{L^2}^2)^{(q-1)/(q-2)}} \;=\; \frac{1}{c_0^2\,c_1^2\,\|u_0\|_{L^2}^2}; \qquad (21)$$

$$\lim_{q\to\infty}\ln\frac{U}{U_+} \;=\; 0; \qquad (22)$$

$$\liminf_{q\to\infty}\frac{\|u\|_{L^{3q}}^{q}}{\|u\|_{L^{3q-6}}^{q-1}} \;\geqslant\; \|u\|_{L^\infty}. \qquad (23)$$

References

- Chuong V. Tran and XY, *A geometrical regularity criterion in terms of velocity profiles for the three-dimensional Navier-Stokes equations*. Q. Jl Mech. Appl. Math., Vol 72, No. 4, 545–562.

Thank You for Your Attention!