A Nontraditional Regularity Criterion for the 3D Navier-Stokes Equations

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Plan of the Talk

1. Introduction
   a) The incompressible Navier-Stokes equations;
   b) The well-posedness problem;
   c) Regularity criteria.


3. Proof of Main Result.
   a) Pressure moderation;
   b) Energy estimate;
   c) Instantaneous decay;
   d) Proof of the main result.
Introduction: The Navier-Stokes Equations

- 3D NSE on $\mathbb{R}^3 \times [0, T)$.

\[
\begin{align*}
  u_t + (u \cdot \nabla)u &= -\nabla p + \nu \Delta u, \\
  \nabla \cdot u &= 0, \\
  u(x, 0) &= u_0(x).
\end{align*}
\]  

$x \in \mathbb{R}^3, t \in [0, T)$.  

- Will set $\nu = 1$ in the following.  

- Leray 1934.
  
  - For $u_0 \in L^2$ there is a weak solution $u \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$ for any $T > 0$.  
  
  - For $u_0 \in H^1 \cap C^1$ there is $T_0 > 0$ and a unique classical solution on $[0, T_0)$.  

Introduction: The Well-posedness Problem

• Many improvements and generalizations (Kato 1972, Caffarelli-Kohn-Nirenberg 1982, Giga 1986, ...).

• Basic situation remains:
  ○ Weak solutions exist globally in time with uniqueness unknown;
  ○ Classical solutions are unique but their global existence has not been proved.

• The questions.
  ○ Are weak solutions unique?
  ○ Does smooth solution exist globally in time?

• Global regularity only established under additional hypotheses (regularity criteria).
**Introduction: Classical Regularity Criteria**

- **Prodi-Serrin-Ladyzhenskaya**: Weak solutions are smooth on \((0, T]\) (and thus can be extended beyond \(T\)) under the following additional assumption:

\[
\int_0^T \|u(t)\|_{L^s}^r \, dt < \infty, \quad s \in [3, \infty], \ r > 0, \ \frac{3}{s} + \frac{2}{r} \leq 1.
\]  

\(s = 3\): Escauriaza-Seregin-Sverak 2003;

- **PSL-type criteria for pressure**:
  
  - Observation: \(\nabla \cdot [u_t + u \cdot \nabla u + \nabla p - \nu \Delta u] = 0 \implies -\Delta p = \nabla \cdot [\nabla \cdot (u \otimes u)] \implies p \sim u^2.\)  

\(J=5\)

- **Calderon-Zygmund inequality**: For \(\gamma \in (1, \infty)\),

\[
\|p\|_{L^r} \lesssim \|u^2\|_{L^r} = \|u\|_{L^{2\gamma}}^2.
\]

\(J=6\)

- Expect criteria of the form \(p \in L'(0, T; L^s)\) with \(s > \frac{3}{2}\) and \(\frac{3}{s} + \frac{2}{r} \leq 2\). (Chae-Lee 2001, Berselli 2002)
Introduction: Other Regularity Criteria

- Generalizations of PSL.
  - Weaken by a power factor $(1 + ||u||)^{-\kappa}$. (Tran-Yu 2017)
  - Weaken by replacing the full vector $u$ with one component. (Kukavica-Ziane 2006, Zhou-Pokorny 2010, Cao-Titi 2011, Chemin-Zhang 2017, Han-Lei-Li-Zhao 2018, ...)
  - And many more.

Main Result: Statement

**Theorem 1.** Let $u$ solve the NSE and be smooth up to (not including) $T > 0$. Assume

i. $u_0 \in L^2 \cap L^\infty$;

ii. $\exists Q(t) \in L^{\infty}_{\text{loc}}[0, T), \forall q > Q(t), \exists f_q(x, t) \in W^{1, \infty}$ and $u \cdot \nabla f_q = 0$, such that $D(x, t) := \nabla \cdot ((p + f_q)|u|^{q-2}u) \neq 0$ in $\Omega(t, q) := \{(x, t)| |u| > U(t, q)\}$ where $U(t, q) = C(q) \|u\|_{L^{3q-6}}$.\(^1\)

Then $u$ remains smooth beyond $T$.

**Remark 2.**

- $Q(t)$ does not have to remain bounded as $t \nearrow T$. For example it could be $\frac{1}{T-t}$.
- No assumption on any of the norms of $u$ or $p$.

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\(^1\) The constant $C(q)$, independent of $t$, will be defined later.
Figure 1. A schematic description of the high velocity region $\Omega(t, q)$. Note that at a fixed time $t$, the locations of $u \cdot \nabla p = 0$ and $u \cdot \nabla |u| = 0$ are fixed but $\Omega(t, q)$ is shrinking as $q \to \infty$. 
Main Result: A Paradigm Scenario

- Assume that the streamlines remain laminar and smooth inside $\Omega(t, q)$;
- $d(\Omega)$: arc length of the longest streamline within $\Omega$;
- $u \cdot \nabla |u| = 0$ and $u \cdot \nabla p = 0$ are smooth enough surfaces, and do not intersect.

**Corollary 3.** Under these assumptions, if $q^{1/2} d(\Omega(t, q)) \to 0$ as $q \to \infty$ at every time $t \in (0, T)$, then the solution remains smooth beyond $T$.

- The convergence $q^{1/2} d(\Omega(t, q)) \to 0$ does not need to be uniform in $t$.

**Proof.** Observe $D = |u|^{q-1} \left[ u \cdot \nabla p + (q - 2) (p + f) \frac{u \cdot \nabla |u|}{|u|} \right]$.

1. Take $f_q \equiv f$ where $u \cdot \nabla f = 0$ and $f = -p$ along $u \cdot \nabla |u| = 0$;
2. $(q - 2) (p + f) \frac{u \cdot \nabla |u|}{|u|} \leq c (q - 2) d(\Omega)^2$ in $\Omega(t, q)$;
3. $|u \cdot \nabla p| \geq C$ in $\Omega(t, q)$.

Thus $D \neq 0$ for all $q$ large enough.
Proof: Pressure Moderation

Lemma 4. Let \( u: \mathbb{R}^3 \mapsto \mathbb{R}^3, f: \mathbb{R}^3 \mapsto \mathbb{R}, \) and \( g: \mathbb{R}^+ \cup \{0\} \mapsto \mathbb{R}. \) Assume

i. \( \nabla \cdot u = 0, u \cdot \nabla f = 0; \)

ii. \( u \in H^1 \cap L^2 \cap L^\infty, f \in W^{1,\infty}, \) and \( g \in L^\infty \cap L_{\text{loc}}. \)

Then for arbitrary \( q \in \mathbb{N}, \) we have

\[
\int_{\mathbb{R}^3} f(x,t) g(|u|) |u|^q u \cdot \nabla |u| \, dx = 0. \tag{7}
\]

Remark 5. We call \( \mathcal{P}(x,t) := f(x,t) g(|u|) \) a “pressure moderator”.

Proof.

1. Let \( H(s) := \int_0^s g(r) r^q \, dr; \) Need to show \( u \cdot \nabla H(|u|) = g(|u|) |u|^q u \cdot \nabla |u|. \)

2. Check:

   i. \( H(s) \) is Lipschitz;

   ii. \( H(|u|) \in L^2. \)

3. Apply Theorem 2.1.11 of W. P. Ziemer Weakly Differentiable Functions.
Proof: Energy Estimate

• Integration of the momentum equation.

\[
\frac{1}{q} \frac{d}{dt} \|u\|_{L^q}^q = (q-2) \int_{\mathbb{R}^3} p |u|^{q-3} \cdot \nabla |u| \, dx \\
- (q-2) \| |u|^{(q-2)/2} \nabla |u| \|_{L^2}^2 - \| |u|^{(q-2)/2} \nabla u \|_{L^2}^2.
\] (8)

• Localized pressure moderator (The cut-off $U$ will be chosen later)

\[
g(s) = \begin{cases} 
0 & s \leq U \\
1 & s > U
\end{cases} \Rightarrow \mathcal{P}(x,t) := f(x,t)g(|u|) = 0 \text{ outside } \Omega := \{|u| > U\}.
\] (9)

• (8) becomes

\[
\frac{1}{q} \frac{d}{dt} \|u\|_{L^q}^q = (q-2) \int_{\mathbb{R}^3} (p + \mathcal{P}) |u|^{q-3} \cdot \nabla |u| \, dx \\
- (q-2) \| |u|^{(q-2)/2} \nabla |u| \|_{L^2}^2 - \| |u|^{(q-2)/2} \nabla u \|_{L^2}^2 \\
= (q-2) \int_\Omega (p + f) |u|^{q-3} \cdot \nabla |u| \, dx \\
+ (q-2) \int_\Omega \mathcal{P} |u|^{q-3} \cdot \nabla |u| \, dx \\
- (q-2) \| |u|^{(q-2)/2} \nabla |u| \|_{L^2}^2 - \| |u|^{(q-2)/2} \nabla u \|_{L^2}^2.
\] (10)
Proof: Contribution from $\Omega^c$

- Take

$$U = U(t, q) := C(q) \|u\|_{L^{3q-6}}, \quad C(q) = \left(c_0 c_1 q \|u_0\|_{L^2}\right)^{-2/(q-2)} \quad (11)$$

where $c_0, c_1$ come from the Calderon-Zygmund and Sobolev inequalities

$$\|p\|_{L^2} \leq c_0 \|u\|_{L^3}^2, \quad \|h\|_{L^6} \leq c_1 \|\nabla h\|_{L^2}. \quad (12)$$

- Set $\Omega = \Omega(t, q) := \{(x, t) \mid |u| > U\}$.

- Contribution outside $\Omega$ is then negligible.

$$\int_{\Omega^c} p \ |u|^{q-3} u \cdot \nabla |u| \ dx \leq \frac{1}{2} \|u\|^{(q-2)/2} \|
abla |u| \|_{L^2}. \quad (13)$$

Proof. Standard interpolation, etc.
Proof: Main Estimate

• Re-formulation.
\[
\int_{\Omega} p |u|^{q-3} u \cdot \nabla |u| \, dx = \int_{\Omega} (p + f) |u|^{q-2} u \cdot \frac{\nabla (|u|/U)}{|u|/U} \, dx \\
= \int_{\Omega} \ln\left(\frac{U}{|u|}\right) \nabla \cdot [(p + f) |u|^{q-2} u] \, dx. \quad (14)
\]

• Dissipation terms.
\[
||u|^{(q-2)/2} \nabla u||_{L^2}, ||u|^{(q-2)/2} \nabla |u||_{L^2} \geq \frac{2}{q c_1} ||u||^{q/2}_{L^{3q}}. \quad (15)
\]

• Main Estimate.
\[
\frac{d}{dt} ||u||^{q}_{L^q} \leq q (q - 2) \int_{\Omega} \ln\left(\frac{U}{|u|}\right) D(x, t) \, dx - \frac{2}{c_1^2} ||u||^{q}_{L^{3q}}. \quad (16)
\]

where
\[
D(x, t) := \nabla \cdot [(p + f) |u|^{q-2} u]. \quad (17)
\]

• Strategy: RHS of (16) <0 for all q > Q(t).

• Problem: Q(t) may be unbounded as t \to \infty.
Proof: Instantaneous Decay

Lemma 6. Let $u$ solve the 3D Navier-Stokes equations that is smooth on $(0, T)$. Assume

i. $u_0 \in L^2 \cap L^\infty$;

ii. $\exists Q(t) \in L^\infty_{loc}[0, T), \forall t \in (0, T), \forall q > Q(t), \frac{d}{dt} \|u\|_{L^q} < 0$.

Then the solution remains smooth beyond $T$.

**Proof.** Wlog $\|u_0\|_{L^2} \leq 1$. Assume $u$ “blows up” at $T$.

1. Interpolation $\Rightarrow \forall q \in [2, \infty], \|u_0\|_{L^q} \leq M$ for some constant $M$;
2. Prodi-Serrin $\Rightarrow \exists q_0 > 3, \|u\|_{L^{q_0}} \to \infty$ at $T$;
3. Blow-up rate: $\|u\|_{L^{q_0}} \geq \frac{1}{(T-t)\beta}$ for $t \in (T - \delta, T)$;
4. Interpolation $\Rightarrow \forall q > q_0, \|u\|_{L^q} \geq \frac{1}{(T-t)\beta}$;
5. But $\|u\|_{L^q} \leq \|u_0\|_{L^q} \leq M$ until $Q(t) > q$. Contradiction for large $q$.

**Remark 7.** There may be no single $q$ such that $\frac{d}{dt} \|u\|_{L^q} < 0$ for all $t \in (0, T)$.
Proof of the Theorem

- Recall the main estimate (16).

\[
\frac{d}{dt} \|u\|_{L^q}^q \leq q(q-2) \int_{\Omega} \ln \left( \frac{U}{|u|} \right) D(x, t) \, dx - \frac{2}{c_1^2} \|u\|_{L^3}^q. 
\]  

(18)

- Strategy: Fix \( t \in (0, T) \). Show RHS < 0 as \( q \to \infty \).

- Assumption \( D \neq 0 \) in \( \Omega \). Thus \( D > 0 \) or \( D < 0 \).
  
  - \( D > 0 \) \( \implies \) RHS < 0;
  
  - Only need to work on \( D < 0 \).
Proof of the Theorem: The case $D < 0$

1. Mean value theorem. $U_+ \in [U(t,q), \|u\|_{L^\infty}]$.

\[
\int_\Omega \ln\left(\frac{U}{|u|}\right) D(x,t) \, dx = \ln \frac{U}{U_+} \int_\Omega \nabla \cdot [(p+f)|u|^{q-2}u] \, dx
\]

\[= U^{q-1} \ln \frac{U}{U_+} \int_{\partial\Omega} (p+f) \hat{u} \cdot dA. \tag{19}\]

2. Substitute in $U = (c_0 c_1 q \|u_0\|_{L^2})^{-2/(q-2)} \|u\|_{L^{3q-6}}$.

\[
\text{RHS} \leq \left[ \frac{q(q-2)}{(c_0^2 c_1^2 q^2 \|u_0\|_{L^2}^2)^{(q-1)/(q-2)}} \ln \frac{U}{U_+} \int_{\partial\Omega} \cdots dA - \frac{2}{c_1^2} \frac{\|u\|_{L^{3q}}^q}{\|u\|_{L^{3q-6}}^{q-1}} \right] \|u\|_{L^{3q-6}}^{q-1} \tag{20}\]

3. $q \not\to \infty$: Note that $t$ is fixed here.

\[
\lim_{q \to \infty} \frac{q(q-2)}{(c_0^2 c_1^2 q^2 \|u_0\|_{L^2}^2)^{(q-1)/(q-2)}} = \frac{1}{c_0^2 c_1^2 \|u_0\|_{L^2}^2}; \tag{21}\]

\[
\lim_{q \to \infty} \ln \frac{U}{U_+} = 0; \tag{22}\]

\[
\liminf_{q \to \infty} \frac{\|u\|_{L^{3q}}^q}{\|u\|_{L^{3q-6}}^{q-1}} \geq \|u\|_{L^\infty}. \tag{23}\]
Thank You for Your Attention!