# Some Problems in General Relativity and Memories of Caltech during 1993-2005

International Workshop on Recent Developments in Applied Mathematics and its Applications

California Institute of Technology

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# My Postdoctoral Stay at Caltech: 1993–1997



## Applied Math (AMath) Faculty:

- Don Cohen
- Joel Franklin
- Herb Keller (primary mentor)
- Dan Meiron
- Phil Saffman
- Gerald Whitham
- Tom Hou (1993–)
- Oscar Bruno (1995–)

## AMath Research Scientist/Postdoc:

- Eric Van de Velde (CRPC Fac)
- Stephan Vandewalle (ofc mate)
- + von Karman instructors

## Other Caltech Faculty:

- Kip Thorne (physics)
  - Eanna Flanagan
  - Eric Poisson
  - Scott Hughes, ...
- Bill Goddard (chemistry)
- Jerry Marsden (CDS; 1992–)
- Peter Schroeder (CS; 1995–)
- + many others

## AMath visitors:

- Edris Titi
- + many others

## AMath students I remember:

- Many who are here today...

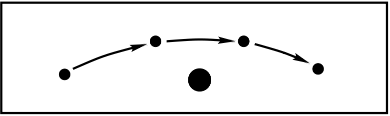
## Herb's three commands to me the day I arrived at Caltech:

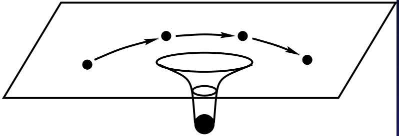
- "Lose the Tie"
- "Take Kip Thorne's GR Class (Phys 236) and teach me GR"
- "Start biking if you want to write joint papers with me"

# General Relativity

Einstein's 1915 **general theory of relativity** states what we experience as gravity is simply the **curvature of our spacetime** when it is viewed as a geometrical object $\mathcal{M}$, known as a *pseudo-Riemannian manifold*.

Newtonian vs. General Relativistic Theories of Gravity:

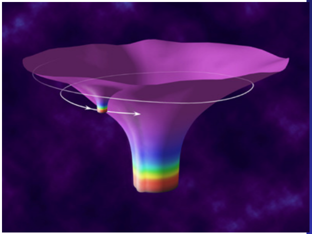




Curvature in our **spacetime** $\mathcal{M}$ is governed by the **Einstein Equations**.

Einstein Equations also predict that accelerating masses produce **gravitational waves**, perturbations in the **metric tensor** of $\mathcal{M}$.

(Image borrowed from LIGO website.)



# Einstein Equations

The Einstein equations are a system of differential equations describing how spacetime curves in response to matter and energy.

Geometric piece of the equations can be understood by examining how derivatives in calculus must be modified when space (or spacetime) is curved:

$$\text{Flat:} \quad V^a{}_{,bc} - V^a{}_{,cb} = 0, \qquad V^a{}_{,b} = \frac{\partial V^a}{\partial x^b}.$$

$$\text{Curved:} \; V^a{}_{;bc} - V^a{}_{;cb} = R^a{}_{dbc} V^d, \quad V^a{}_{;b} = V^a{}_{,b} + \Gamma^a_{bc} V^c.$$

Let us note what form $R^a{}_{dbc}$ takes, and give names to some objects:

- $R^a{}_{dbc} = \Gamma^a_{bd,c} - \Gamma^a_{cd,b} + \Gamma^a_{ec}\Gamma^e_{bd} - \Gamma^a_{eb}\Gamma^e_{cd}$; Riemann tensor
- $R_{ab} = R_{acb}{}^c$, $\quad R = R_a{}^a$; Ricci tensor, scalar curv.
- $G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}$; Einstein tensor
- $T_{ab}$; Stress-energy tensor

The Einstein equations relate the mathematical object ($G_{ab}$) describing curvature of spacetime $\mathcal{M}$ to the mathematical object ($T_{ab}$) representing matter and energy content of our spacetime:

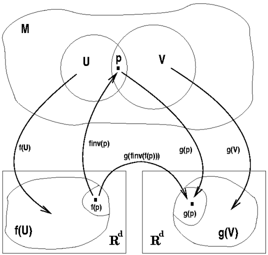
$$G_{ab} = \kappa T_{ab}, \qquad 0 \leqslant a \leqslant b \leqslant 3, \quad \kappa = 8\pi G/c^4. \quad \text{(10 equations)}$$

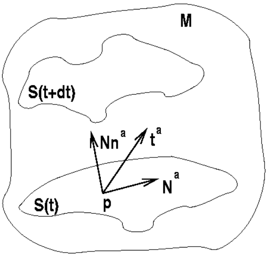
# Reformulation as an Initial-Value Problem

Space and time are all mixed up in Einstein's Equation: $G_{ab} = \kappa T_{ab}$.

Hope for reformulation as well-posed **initial-value problem**; "future" would then be determined by solution of a time-dependent differential equation for the "metric" of space at any future time.

This program was completed by Yvonne Choquet-Bruhat and other mathematicians by the 1950's; the famous book of Hawking & Ellis in 1973 summarizes this theory. (Summarized in MTW, Wald, etc.)




Result: 12-component evolution system for $(\hat{h}_{ab}, \hat{k}_{ab})$ on foliation $S(t)$:

$$\partial_t \hat{h}_{ab} = \boxed{\text{eqn for 3-metric } \hat{h}}, \qquad \partial_t \hat{k}_{ab} = \boxed{\text{eqn for extrinsic curvature } \hat{k}}$$

# Einstein Constraints and Conformal Method

Constrained by 4 coupled eqns on a spacelike hypersurface $\mathcal{M} = S(t)$, with $\hat{\tau} = \hat{k}_{ab}\hat{h}^{ab}$,

$$^3\hat{R} + \hat{\tau}^2 - \hat{k}_{ab}\hat{k}^{ab} - 2\kappa\hat{\rho} = 0, \qquad \hat{\nabla}^a\hat{\tau} - \hat{\nabla}_b\hat{k}^{ab} - \kappa\hat{j}^a = 0.$$

*Lichnerowicz-York conformal decomposition:* split initial data into 8 freely specifiable pieces plus 4 determined via: $\hat{h}_{ab} = \phi^4 h_{ab}$, $\hat{\tau} = \hat{k}_{ab}\hat{h}^{ab} = \tau$, and

$$\hat{k}_{ab} = \phi^{-10}[\sigma^{ab} + (\mathcal{L}w)^{ab}] + \frac{1}{4}\phi^{-4}\tau h^{ab}, \quad \hat{j}^a = \phi^{-10}j^a, \quad \hat{\rho} = \phi^{-8}\rho.$$

Produces coupled elliptic system for conformal factor $\phi$ and a $w^a$:

$$\boxed{\begin{aligned} -8\Delta\phi + R\phi + \frac{2}{3}\tau^2\phi^5 - (\sigma_{ab} + (\mathcal{L}w)_{ab})^2\phi^{-7} - 2\kappa\rho\phi^{-3} &= 0, \\ -\nabla_a(\mathcal{L}w)^{ab} + \frac{2}{3}\phi^6\nabla^b\tau + \kappa j^b &= 0. \end{aligned}}$$

Differential structure on $\mathcal{M}$ defined through background 3-metric $h_{ab}$:

$$(\mathcal{L}w)^{ab} = \nabla^a w^b + \nabla^b w^a - \frac{2}{3}(\nabla_c w^c)h^{ab}, \qquad \nabla_b V^a = V^a{}_{;b} = V^a{}_{,b} + \Gamma^a_{bc}V^c,$$

$$V^a{}_{,b} = \frac{\partial V^a}{\partial x^b}, \qquad \Gamma^a_{bc} = \frac{1}{2}h^{ad}\left(\frac{\partial h_{db}}{\partial x^c} + \frac{\partial h_{dc}}{\partial x^b} - \frac{\partial h_{bc}}{\partial x^d}\right). \qquad (\Gamma^a_{bc} = \Gamma^a_{cb})$$

# The Conformal Method as an Elliptic System

Let $\mathcal{M}$ be a space-like Riemannian 3-manifold with (possibly empty) boundary submanifold $\partial\mathcal{M}$, split into disjoint submanifolds satisfying:

$$\partial_D\mathcal{M} \cup \partial_N\mathcal{M} = \partial\mathcal{M}, \qquad \partial_D\mathcal{M} \cap \partial_N\mathcal{M} = \varnothing. \qquad (\overline{\partial_D\mathcal{M}} \cap \overline{\partial_N\mathcal{M}} = \varnothing)$$

Metric $h_{ab}$ associated with $\mathcal{M}$ induces boundary metric $\sigma_{ab}$, giving boundary value formulation of conformal method for $\phi$ and $w^a$:

$$\boxed{\begin{array}{lll} L\phi + F(\phi, w) = 0, \ \text{in } \mathcal{M}, & & \text{(Hamiltonian)} \\ \mathbb{L}w + \mathbb{F}(\phi) = 0, \ \text{in } \mathcal{M}, & & \text{(Momentum)} \\ (\mathcal{L}w)^{ab}\nu_b + C^a{}_b w^b = V^a_\phi \ \text{on } \partial_N\mathcal{M}, & \textit{and} & w^a = w^a_D \ \text{on } \partial_D\mathcal{M}, \\ (\nabla^a\phi)\nu_a + k_w(\phi) = g \ \text{on } \partial_N\mathcal{M}, & \textit{and} & \phi = \phi_D \ \text{on } \partial_D\mathcal{M}, \end{array}}$$

*where:*

$$L\phi = -\Delta\phi, \quad (\mathbb{L}w)^a = -\nabla_b(\mathcal{L}w)^{ab},$$

$$F(\phi, w) = a_R\phi + a_\tau\phi^5 - a_w\phi^{-7} - a_\rho\phi^{-3}, \quad \mathbb{F}(\phi) = b^b_\tau\phi^6 + b^b_j,$$

*with:*

$$a_R = \frac{R}{8}, \ a_\tau = \frac{\tau^2}{12}, \ a_w = \frac{1}{8}[\sigma_{ab} + (\mathcal{L}w)_{ab}]^2, \ a_\rho = \frac{\kappa\rho}{4}, \ b^b_\tau = \frac{2}{3}\nabla^b\tau, \ b^b_j = \kappa j^b,$$

$$(\mathcal{L}w)^{ab} = \nabla^a w^b + \nabla^b w^a - \frac{2}{3}(\nabla_c w^c)h^{ab}, \qquad \nabla_b V^a = V^a{}_{;b} = V^a{}_{,b} + \Gamma^a_{bc}V^c,$$

$$V^a{}_{,b} = \frac{\partial V^a}{\partial x^b}, \qquad \Gamma^a_{bc} = \frac{1}{2}h^{ad}\left(\frac{\partial h_{db}}{\partial x^c} + \frac{\partial h_{dc}}{\partial x^b} - \frac{\partial h_{bc}}{\partial x^d}\right). \qquad (\Gamma^a_{bc} = \Gamma^a_{cb})$$

# The Reality of finding something I could work on in GR...

- Understanding enough GR to work on math questions: daunting.
- It became clear this was going to be a much longer term project.
- Herb had a gut feeling that the nonlinear structure in GR would need analytic and numerical bifurcation analysis.
  (He turned out to be right, but not possible until well after 2008.)
- Simultaneously I pursued other things with Caltech visitors; e.g.:
  - M. H. and S. Vandewalle, *Schwarz methods: To symmetrize or not to symmetrize*, SIAM J. Numer. Anal., 34 (1997), pp. 699–722.
  - M. H. and E. Titi, *Determining projections and functionals for weak solutions of the Navier-Stokes equations*, in Recent Developments in Optimization Theory and Nonlinear Analysis, Y. Censor and S. Reich, eds., vol. 204 of Contemporary Mathematics, Providence, RI, 1997, American Mathematical Society, pp. 125–138.
- Meanwhile, Kip Thorne generously invited me to sit in his weekly research group meetings (1995-1997).
- AMath Courses I taught: 204A (Fall 1995), 105A (Winter 1996), 105B (Spring 1996), 104 (Fall 1996), 220C (Spring 1997).
- My one-off paper with E. Titi above led to my first job as an assistant professor at UC Irvine (1997-1998).

# Away from Caltech: 1998–2002

## UC Irvine Period: 1997-1998

- Recruited by UC Irvine Math to help build Applied Math Group.
- Worked with E. Titi on recruiting others for coming year.
- Stayed in contact with Caltech AMath & Thorne Group.
- Extended my Caltech AMA 204a notes for grad course at UCI.
- Found my first two graduate students (later moved to UCSD).
- Recruited hard by UCSD Math in Spring 1998.
- Final Act before moving to UCSD in Summer 1998: helped recruit Hongkai Zhao (Szego at Stanford), who then really started a buildup in UCI applied math.
- Remained in contact with E. Titi; sent him my UCSD undergraduate honors student (Evelyn Lunasin) as his PhD student, and she later rejoined me at UCSD as a postdoc to work on determining degrees of freedom in Navier-Stokes models:
    - M. Holst, E. Lunasin, and G. Tsogtgerel, *Analysis of a general family of regularized Navier-Stokes and MHD models*, J. Nonlin. Sci., 20 (2010), pp. 523–567.

# Away from Caltech: 1998–2002

## UC San Diego Period: 1998-2002

- Summer 1998: Arrived at UCSD (Spouse Mai in LA until 1999).
- June 1998: Frantically wrote first NSF Proposal (CAREER).
  - Proposal centered around Einstein Constraint equations: approximation theory and adaptive algorithms.
- Some papers produced in this direction:
  - R. Bank and M. Holst, *A new paradigm for parallel adaptive mesh refinement*, SIAM Rev., 45 (2003), pp. 291–323.
  - M. Holst, *Applications of domain decomposition and partition of unity methods in physics and geometry (plenary paper)*, Domain Decomposition Methods in Science and Engineering, Mexico City, Mexico, June 2003, UNAM, pp. 63–78.
  - M. Holst, *Adaptive numerical treatment of elliptic systems on manifolds*, Adv. Comput. Math., 15 (2001), pp. 139–191.
- AiCM Paper above: Derived *a priori* and *a posteriori* FE error estimates for Einstein constraints, based on solvability and regularity assumptions.
- Robert Bartnik: generously read article and made helpful comments; outlined large gaps in existing PDE theory for constraints.
- Spring 2002: Kip Thorne drove down to UCSD to invite me to help organize a 2-year visitor program at Caltech to start in Fall 2002.

# LIGO: Laser Interferometer Gravitational-wave Observatory

LIGO is one of several recently constructed gravitational detectors.

Design of LIGO based on measuring distance changes between objects in orthogonal directions as metric tensor ripple propagates through device.

Two L-shaped LIGO detectors (Washington & Louisiana), 1.5m $\times$ 4km, have phenomenal sensitivity, on order $10^{-15}$m to $10^{-18}$m.

**Fact:** LIGO turns on in 2005; hinges on simulation & analysis pipeline.

**Problem:** LIGO funding initiated in 1992; by 2002 no real progress on codes that could simulate even a single orbit of a pure binary system.

**Kip's Plan:** Bring together best math & physics people to sort it out.

# My Second Stay at Caltech: 2003–2005

- I was one of 12 organizers of the Visitor program (11 physicists).
- I was at Caltech during much of the next three academic years:
  - 2002-2003: Program first year; lived in Pasadena with Herb Keller.
  - 2003-2004: Based back at UCSD (1st child Maven born May 2003).
  - 2004-2005: Lived in Hawking house on Catalina (w/ Mai & Maven).
    (2nd child Makenna born October 2005.)

ACM (name change!) Faculty:

- Oscar Bruno
- Tom Hou
- Herb Keller
- Dan Meiron
- Niles Pierce (1999–)
- Emmanuel Candes (2000–)
- Houman Owhadi (2003–)

Other Caltech Faculty:

- Kip Thorne (physics)
- Jerry Marsden (CDS)
  - Melvin Leok (PhD 2004)
  - Ari Stern (PhD 2009)
- + others

Caltech Physics people:

- Lee Lindblom, Mark Sheel
- Harold Pfeiffer, Franz Pretorius
- + others

Math Physics visitors:

- Oscar Reula
  - Gabriel Nagy
  - Manuel Tiglio
- Olivier Sarbach, + others

Math visitors:

- Doug Arnold, Robert Bartnik
- Jim Isenberg, Vince Moncrief
- Niall O'Murchadha, S.T. Yau
- + others

# Visitor Program: Great Success

- Each quarter of the program focused on different issues:
  - Weakly, strongly, and symmetric hyperbolic formulations of GR.
  - Understanding the properties of the constraint equations.
  - Understanding interior and exterior boundary conditions.
  - Coordinates, constraint control, + many other obstacles.
- By 2005: First working code presented at BANFF/BIRS by Franz Pretorious – tens of orbits of binary black hole system, inspiral and merger, ring-down, and gravitational wave extraction! This was accomplished by combining various advanced techniques from applied and computational math community, such as:
  - Well-posed hyperbolic formulation (had been much confusion)
  - Local AMR (driven by *a posteriori* indicators)
  - Inner boundary condition models (black hole isometry models)
  - Exterior boundary condition models (radiation conditions)
  - Constraint control; one of our contributions in this direction:
    - M. Holst, L. Lindblom, R. Owen, H. Pfeiffer, M. Scheel, and L. Kidder, *Optimal constraint projection for hyperbolic evolution systems*, Phys. Rev. D, 70 (2004), pp. 84017(1)–84017(17).
- Within a year, multiple groups had produced similar working codes using different core techniques such as spectral methods.
- LIGO came online in 2005; codes were ready just in time. For next decade (2005-2015), simulation pipeline was continually refined.
- *Math community contributed greatly to the success of the GR simulation part of the LIGO pipeline, and this is not widely known.*

# LIGO Detection in Fall 2015

In February 2016, NSF announced an upcoming conference press for Feb 11 with the humble title *"Scientists to Provide Update on the Search for Gravitational Waves"*. It would not take place in some dusty lab, but rather at the National Press Club in Washington, DC.

When the press conference began, the LIGO Laboratory director David Reitze simply announced: *"Ladies and gentlemen. We have detected gravitational waves. We did it."*

On 14 September 2015, both LIGO devices nearly simultaneously detected a *clear, loud, and violent inspiral, collision, merger, and ringdown of a binary black hole pair*, each of which had a solar mass in range 10-50, with roughly equivalent of three solar masses in energy released as gravitational radiation. Radiation traveled outward from at speed of light, reaching LIGO detectors roughly 1.3 billion years later.

The detection was only possibly through the use of modern data analysis techniques combined with computer simulations of wave emission from this type of binary collision, produced through very *detailed numerical simulations of the full Einstein equations*.

# LIGO First Detection in 2015 and Nobel Prize in 2017

We documented the involvement of the mathematics community behind the scenes of LIGO, including the Caltech Visitor Program, in a 2016 AMS Bulletin Article that appeared just after the first gravitational wave detection by LIGO was announced by NSF.

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 53, Number 4, October 2016, Pages 513–554
http://dx.doi.org/10.1090/bull/1544
Article electronically published on August 2, 2016

## THE EMERGENCE OF GRAVITATIONAL WAVE SCIENCE: 100 YEARS OF DEVELOPMENT OF MATHEMATICAL THEORY, DETECTORS, NUMERICAL ALGORITHMS, AND DATA ANALYSIS TOOLS

MICHAEL HOLST, OLIVIER SARBACH, MANUEL TIGLIO, AND MICHELE VALLISNERI

*In memory of Sergio Dain*

Abstract. On September 14, 2015, the newly upgraded Laser Interferometer Gravitational-wave Observatory (LIGO) recorded a loud gravitational-wave (GW) signal, emitted a billion light-years away by a coalescing binary of two stellar-mass black holes. The detection was announced in February 2016, in time for the hundredth anniversary of Einstein's prediction of GWs within the theory of general relativity (GR). The signal represents the first direct detection of GWs, the first observation of a black-hole binary, and the first test of GR in its strong-field, high-velocity, nonlinear regime. In the remainder of its first observing run, LIGO observed two more signals from black-hole binaries, one moderately loud, another at the boundary of statistical significance. The detections mark the end of a decades-long quest and the beginning of GW astronomy: finally, we are able to probe the unseen, electromagnetically dark Universe by *listening* to it. In this article, we present a short historical overview of *GW science*: this young discipline combines GR, arguably the crowning achievement of classical physics, with record-setting, ultra-low-noise laser interferometry, and with some of the most powerful developments in the theory of differential geometry, partial differential equations, high-performance computation, numerical analysis, signal processing, statistical inference, and

# Period Since Caltech: 2006–Present



Returning to UCSD in 2005, the recent interactions at Caltech led to:

- My numerical analysis research evolving toward two new topics.
- New research direction on analysis of the Einstein constraints.

A conversation with Emmanuel Candes (circa 2004) in part led me to an interest in convergence theory for adaptive algorithms; e.g.

- L. Chen, MH, and J. Xu, *The finite element approximation of the nonlinear Poisson-Boltzmann Equation*, SIAM J. Numer. Anal., 45 (2007), pp. 2298–2320.
- L. Chen, MH, and J. Xu, *Convergence and optimality of adaptive mixed finite element methods*, Math. Comp., 78 (2009), pp. 35–53.
- MH, S. Pollock, and Y. Zhu, *Convergence of goal-oriented adaptive finite element methods for semilinear problems*, Computing and Visualization in Science, 17 (2015), pp. 43–63.
- MH and S. Pollock, *Convergence of goal-oriented adaptive finite element methods for nonsymmetric problems*, Numerical Methods for Partial Differential Equations, 32 (2016), pp. 479–509.
- MH, Y. Li, A. Mihalik, and R. Szypowski, *Convergence and optimality of adaptive mixed methods for Poisson's equation in the FEEC framework*, J. Comput. Math., 38 (2020), pp. 748–767.

# Period Since Caltech: 2006–Present



Conversations with Doug Arnold and Jerry Marsden at Caltech during 2003-2005 led to a new interest in mixed formulations and finite element exterior calculus (FEEC), with applications to geometric PDE; e.g.,


- MH and A. Stern, *Geometric variational crimes: Hilbert complexes, finite element exterior calculus, and problems on hypersurfaces*, Found. Comput. Math., 12 (2012), pp. 263–293.
- MH and A. Stern, *Semilinear mixed problems on Hilbert complexes and their numerical approximation*, Found. Comput. Math., 12 (2012), pp. 363–387.
- A. Gillette, MH, and Y. Zhu, *Finite element exterior calculus for evolution problems*, Journal of Computational Mathematics, 35 (2017), pp. 186–212.
- MH and C. Tiee, *Finite element exterior calculus for parabolic evolution problems on Riemannian hypersurfaces*, Journal of Computational Mathematics, 36 (2018), pp. 792–832.
- MH, Y. Li, A. Mihalik, and R. Szypowski, *Convergence and optimality of adaptive mixed methods for Poisson's equation in the FEEC framework*, J. Comput. Math., 38 (2020), pp. 748–767.

# Einstein Constraint Equations: CMC Case (1944–1995)

Conversations with Jim Isenberg, Vince Moncrief, Robert Bartnik and S.T. Yau over 2003-2005 led me to focus on developing a more complete solution theory for Einstein constraint equations.

Fundamentally there are three cases: CMC, Near-CMC, and Non-CMC

**Case 1:** Constant Mean Curvature (CMC):

$$\nabla^b \tau = 0 \Rightarrow \text{ constraints de-couple.}$$

CMC results during 1944–1995 via exploiting decoupling; reduces to solvability of the Hamiltonian constraint; e.g.

- A. Lichnerowicz, L'integration des équations de la gravitation relativiste et le problème des n corps. *J. Math. Pures Appl.*, 23:37–63, 1944.
- Y. Choquet-Bruhat Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires. *Acta Math.*, 88:141–225, 1952.
- N. Ó. Murchadha and J. York, Initial-value problem of general relativity I. General formulation and physical interpretation. *Phys. Rev. D*, 10(2):428–436, 1974.
- J. Isenberg, Constant mean curvature solution of the Einstein constraint equations on closed manifold. Class. Quantum Grav. **12** (1995), 2249–2274.

# Einstein Constraint Equations: Near-CMC Case (1996–2007)



**Case 2:** Near-CMC:

$$\nabla^b \tau \neq 0 \text{ but } \nabla^b \tau \approx 0 \Rightarrow \text{ constraints weakly coupled.}$$

In 1996, Isenberg-Moncrief show first *near-CMC* existence result:

- J. Isenberg and V. Moncrief, *A set of nonconstant mean curvature solution of the Einstein constraint equations on closed manifolds*, Class. Quantum Grav. **13** (1996), 1819–1847.
- J. Isenberg and J. Park, Asymptotically hyperbolic non-constant mean curvature solutions of the Einstein constraint equations. *Class. Quantum Grav.*, 14:A189–A201, 1997.
- Y. Choquet-Bruhat, J. Isenberg, and J. York, Einstein constraint on asymptotically Euclidean manifolds. *Phys. Rev. D*, 61:084034, 2000.

**Case 3:** Non-CMC:

$$\nabla^b \tau \neq 0 \Rightarrow \text{ constraints coupled arbitrarily strongly.}$$

In 2004 Bartnik and Isenberg state: "Almost nothing known."

- R. Bartnik and J. Isenberg, *The Constraint Equations*, In: Chruściel, P.T., Friedrich, H. (eds) The Einstein Equations and the Large Scale Behavior of Gravitational Fields. Birkhäuser, Basel.

# Einstein Constraint Equations: Non-CMC Case (2008–Present)



**Case 3:** Non-CMC:

$$\nabla^b \tau \neq 0 \Rightarrow \text{ constraints coupled arbitrarily strongly.}$$

Working on this case over the 2005-2008 period, we established the first existence results for the general non-CMC case in:

- MH, G. Nagy, and G. Tsogtgerel, *Far-from-constant mean curvature solutions of Einstein's constraint equations with positive Yamabe metrics*, Phys. Rev. Lett., 100 (2008), pp. 161101.1–161101.4.
- MH, G. Nagy, and G. Tsogtgerel, *Rough solutions of the Einstein constraints on closed manifolds without near-CMC conditions*, Comm. Math. Phys., 288 (2009), pp. 547–613.

More recent existence results with direct relevance to LIGO:

- MH and G. Tsogtgerel, *The Lichnerowicz equation on compact manifolds with boundary*, Class. Quantum Grav., 30 (2013), pp. 1–31.
- MH and C. Meier, *Non-CMC solutions of the Einstein constraint equations on asymptotically Euclidean manifolds with apparent horizon boundaries*, Class. Quantum Grav., 32 (2014), pp. 1–25.
- MH, C. Meier, and G. Tsogtgerel, *Non-CMC solutions of the Einstein constraint equations on compact manifolds with apparent horizon boundaries*, Comm. Math. Phys., 357 (2018), pp. 467–517.

# Einstein Constraint Equations: Folds and Bifurcations

Herb Keller's intuition back in 1993 was right: *The nonlinear structure of the Einstein constraints allows for solutions that exhibit fold and bifurcation phenomena.*

First discovered by LIGO numerical relativists (circa 2004); reported their nonlinear iterative solvers could find either of two distinct solutions in non-CMC case, depending on choice of initial condition for iteration.

Some work was done by Niall O'Murchadha, Harald Pfeiffer, and some of their collaborators to show this numerically in a predictable way.

Our group and several others began to use analytic and/or numerical bifurcation tools to understand the solution behavior in 2011; e.g.,

- MH and V. Kungurtsev, *Numerical bifurcation analysis of conformal formulations of the Einstein constraints*, Phys. Rev. D, 84 (2011), pp. 124038(1)–124038(8).
- MH and C. Meier, *Non-Uniqueness of Solutions to the Conformal Formulation*, Available as arXiv:1210.2156 [gr-qc]
- J. Dilts, MH, T. Kozareva, and D. Maxwell, *Numerical Bifurcation Analysis of the Conformal Method*, Available as arXiv:1710.03201 [gr-qc]

and has led to the development of an alternative to the conformal method known as the "Drift System"; a first analysis appears here:

- MH, D. Maxwell, and R. Mazzeo, *Conformal fields and the structure of the space of solutions of the Einstein constraint equations*, Adv. Theor. Math. Phys., 26 (2022), pp. 1157–1201.